Comments and Addenda

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Bianchi IV metric with electromagnetic field

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An exact solution of the Einstein equations with an electromagnetic field source and Bianchi IV symmetry is described. This solution possesses, in the limiting case of vanishing source tensor, a previously discovered vacuum metric.

There was recently presented an exact vacuum Bianchi IV solution to the Einstein equations.[1] In the course of determining the solution it was found that the particular choice of metric structure (within the overall context of Bianchi IV symmetry) led to an Einstein tensor which was incompatible with a perfect fluid source. Ensuing study of this anomaly included tests of compatibility with other sources. It has been found that a null field may serve as a source.

For a stress-energy tensor

$$T^{\mu\nu}=U\lambda^{\mu}\lambda^{\nu}\,, \tag{1}$$

where

$$\lambda^{\mu}\lambda_{\mu}=\lambda^{\mu}{}_{;\nu}\lambda^{\nu}=0 \tag{2}$$

and $U(t)$ is the energy density, a pair of solutions have been found. The two correspond, respectively, to radiation in the positive and negative σ^1 directions. The term radiation is used here in a purely formal sense to indicate a null field. At a given time the stress-energy tensor is everywhere identical on the hypersurface of homogeneity. It is radiation in the lowest possible mode. There is no energy transport.

The notation, conventions, and pertinent portions of Ref. 1, suitably modified, are utilized. The stress-energy tensor has vanishing trace so the field equations are

$$R_{\mu\nu}=\kappa T_{\mu\nu}\,, \tag{3}$$

with units so chosen that $\kappa=1$. The metric structure is taken to be the same. Four components of the Ricci tensor vanish identically:

$$R_{02}\equiv R_{03}\equiv R_{12}\equiv R_{13}\equiv 0\,, \tag{4}$$

implying that

$$\lambda_2=\lambda_3=0\,. \tag{5}$$

It follows that in the orthonormal frame

$$T^{11}=T^{00}=T_{11}=T_{00}=1\,, \tag{6}$$

$$T^{10}=T^{01}=-T_{10}=-T_{01}=\pm 1\,. \tag{7}$$

All other components vanish.

The field equations are

$$R_{00}=-\frac{d}{dt}\left(\frac{\dot a}{a}+2\frac{\dot b}{b}\right)-\left(\frac{\dot a}{a}\right)^2-2\left(\frac{\dot b}{b}\right)^2-\frac{\dot f^2}{2}=U\,, \tag{8a}$$

$$R_{01}=\frac{2\dot a}{a^2}-\frac{2\dot b}{ab}-\frac{\dot f}{2a}=\mp U\,, \tag{8b}$$

$$R_{11}=\frac{d}{dt}\left(\frac{\dot a}{a}\right)+\frac{\dot a}{a}\left(\frac{\dot a}{a}+2\frac{\dot b}{b}\right)-\frac{5}{2a^2}=U\,, \tag{8c}$$

$$R_{22}=\frac{d}{dt}\left(\frac{\dot b}{b}\right)+\frac{\dot b}{b}\left(\frac{\dot a}{a}+2\frac{\dot b}{b}\right)+\frac{\dot f^2}{2}-\frac{5}{2a^2}=0\,, \tag{8d}$$

$$R_{23}=\frac{d}{dt}\left(\frac{\dot f}{2}\right)+\frac{\dot f}{2}\left(\frac{\dot a}{a}+2\frac{\dot b}{b}\right)-\frac{1}{a^2}=0\,, \tag{8e}$$

$$R_{33}=\frac{d}{dt}\left(\frac{\dot b}{b}\right)+\frac{\dot b}{b}\left(\frac{\dot a}{a}+2\frac{\dot b}{b}\right)-\frac{\dot f^2}{2}-\frac{3}{2a^2}=0\,. \tag{8f}$$

With use of the connection coefficients given in Ref. 1 the covariant divergence of $T^{\mu\nu}$ is readily calculated to be

$$T^{2\nu}{}_{;\nu} = T^{3\nu}{}_{;\nu} \equiv 0 \tag{9a}$$

and

$$T^{0\nu}{}_{;\nu} = T^{1\nu}{}_{;\nu} = \dot{U} + 2U\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{1}{a}\right) = 0\,. \tag{9b}$$

Equation (8e) implies that unless the off-diagonal term involving f is retained, no solution will exist. Then

$$R_{22} - R_{33} = \dot{f}^2 - 1/a^2 = 0\,, \tag{10}$$

with

$$\dot{f} = \pm 1/a\,. \tag{11}$$

Use of Eq. (11) in (8e) provides

$$\dot{b}/b = \pm 1/a\,. \tag{12}$$

The plus-or-minus signs are coupled. Equations (11) and (12) satisfy $R_{22} = R_{33} = 0$ identically.

The remaining field equations are

$$R_{00} = -\frac{\ddot{a}}{a} \pm \frac{2\dot{a}}{a^2} - \frac{5}{2a^2} = U\,, \tag{13a}$$

$$R_{01} = \frac{2\dot{a}}{a^2} \mp \frac{5}{2a^2} = \mp U\,, \tag{13b}$$

$$R_{11} = \frac{\ddot{a}}{a} \pm \frac{2\dot{a}}{a^2} - \frac{5}{2a^2} = U\,. \tag{13c}$$

The plus-or-minus signs on the left-hand sides are coupled. That on the right-hand side depends on the direction of the radiation.

The solution for a, which is *independent of* U, is obtained from

$$R_{11} - R_{00} = 2\ddot{a} = 0\,, \tag{14}$$

$$a = k_1 t + k_2\,. \tag{15}$$

The second constant merely sets the time origin and may be taken equal to zero:

$$a = kt\,. \tag{16}$$

Examination of Eqs. (13) now shows that for consistency the plus-or-minus signs in Eq. (13b) must be coupled. Thus,

$$U = \frac{\pm 2\dot{a} - \frac{5}{2}}{a^2}\,, \tag{17a}$$

$$U = \frac{\pm 2k - \frac{5}{2}}{k^2 t^2}\,, \tag{17b}$$

where the plus sign corresponds to radiation in the positive σ^1 direction and the minus sign to the negative direction. In order that U be positive

$$\pm 2k - \tfrac{5}{2} > 0\,, \tag{18}$$

with

$$k_+ > +\tfrac{5}{4} \tag{19}$$

or

$$k_- < -\tfrac{5}{4}\,, \tag{20}$$

with the same correspondence to positively and negatively directed radiation.

The solution for b is determined from Eq. (12) with the sign such that Eqs. (9b) are satisfied. Firstly

$$\dot{U}/U = -2/t \tag{21}$$

and

$$\dot{a}/a = 1/t\,, \tag{22}$$

which reduces the equations to

$$\dot{b}/b + 1/a = 0\,. \tag{23}$$

Thus, Eq. (12) must be

$$\frac{\dot{b}}{b} = -\frac{1}{a} \tag{24}$$

$$= \pm\frac{1}{kt} \tag{25}$$

and

$$b = b_0(\pm kt)^{\pm 1/k}\,. \tag{26}$$

Also,

$$\dot{f} = \pm\frac{1}{a}\,, \tag{27}$$

$$f = \ln f_0(\pm kt)^{\pm 1/k}\,. \tag{28}$$

Integration constants b_0 and f_0 are scale factors and may be set equal to 1. The complete metric in the synchronous frame is, for $k > 0$,

$$ds^2 = -\sigma^0\sigma^0 + (kt)^2\sigma^1\sigma^1 + t^{2/k}\{[1 + \ln^2(kt)^{1/k}]\sigma^2\sigma^2 + \ln(kt)^{1/k}(\sigma^2\sigma^3 + \sigma^3\sigma^2) + \sigma^3\sigma^3\}\,. \tag{29}$$

For $k = \pm\frac{5}{4}$ the previously obtained vacuum metric results.

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[1]A. Harvey and D. Tsoubelis, Phys. Rev. D 15, 2734 (1977).